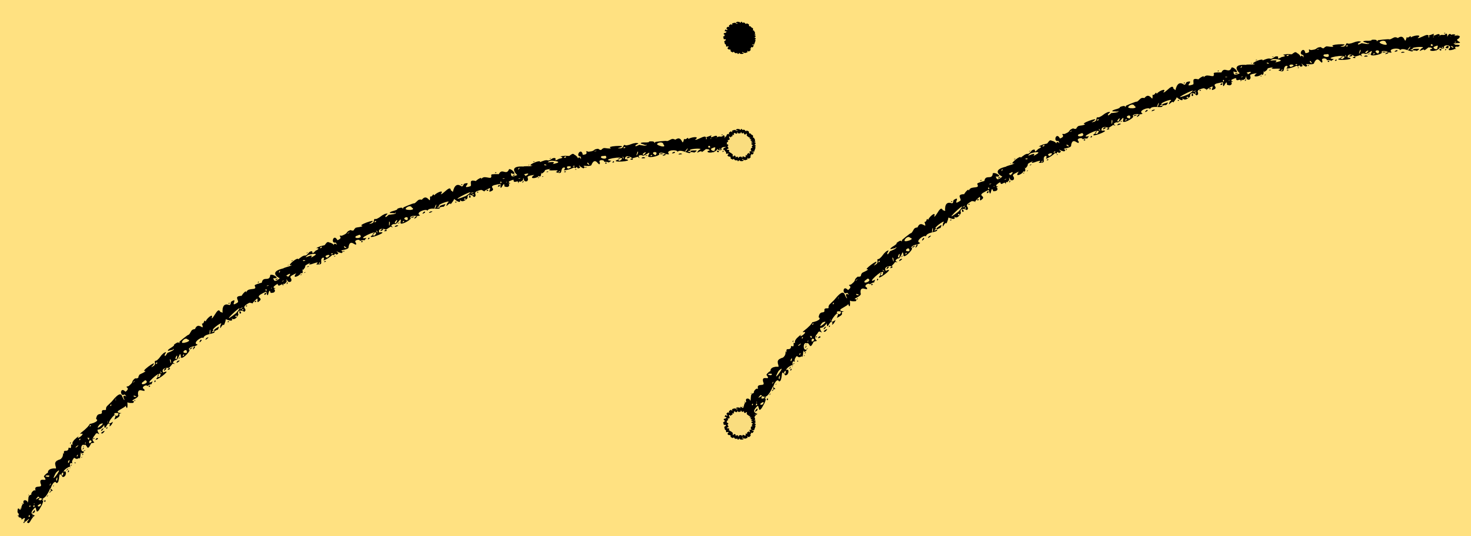
Math 1653W

Wednesday 18 October 2023

dr Adam Abrams

Warm-up: Does this graph have a hole, jump, asymptote, or none of these?



Last time

What does $\lim_{n\to\infty} \frac{n}{n+5} = 1$ mean?

- *Formally:* for any $\varepsilon > 0$,

$$1 - \varepsilon < \frac{n}{n+5} < 1 + \varepsilon \text{ for all } n > \frac{5 - 5\varepsilon}{\varepsilon}.$$

$$\left| \frac{n}{n+5} - 1 \right| < \varepsilon$$

- *Informally:*

$\frac{n}{n+5}$ is very close to 1 when n is very big.

$$\frac{10000}{10005} = 0.9995002\ldots$$

It's possible to make

$$0.9 < \frac{n}{n+5} < 1.1$$

and

$$0.98 < \frac{n}{n+5} < 1.02$$

and

$$0.9999 < \frac{n}{n+5} < 1.0001$$

for all $n > N$, if we choose the right N.

Limit rules

Last time

It will often be useful to know the limit of r^n where r is a constant number.

- If $-1 < r < 1$ then $\lim_{n \to \infty} r^n = 0$.
- If $r = 1$ then $\lim_{n \to \infty} r^n = 1$.
- If $r \le -1$ then $\lim_{n \to \infty} r^n$ does not exist.
- If $r > 1$ then $\lim_{n \to \infty} r^n = \infty$.

Limit rules

When we have a ratio of two polynomials, the limit

$$\lim_{n\to\infty} \frac{An^d + \cdots}{Bn^e + \cdots}$$

can be found very quickly. (Here "$\cdots$" are terms with smaller powers of n).

- If $d < e$ then the limit is 0.
- If $d = e$ then the limit is $\frac{A}{B}$.
- If $d > e$ then
  - the limit is ∞ if $\frac{A}{B} > 0$.
  - the limit is $-\infty$ if $\frac{A}{B} < 0$.

Limit rules

Last time

If the limits all exist and are *finite*, then

- $\lim_{n\to\infty}\left(a_n + b_n\right) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right),$
- $\lim_{n\to\infty}\left(a_n \cdot b_n\right) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$
- $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{x\to a} b_n}$ if $\lim_{n\to\infty} b_n \neq 0,$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ for any real number p.
- $\lim_{n\to\infty}\frac{1}{n} = 0.$

with $b_n = c$ constant

- $\lim_{n\to\infty}\left(c \cdot a_n\right) = c \cdot \left(\lim_{n\to\infty} a_n\right)$

It is often helpful to think of

$$\infty - 5 = \infty, \qquad \frac{\infty}{2} = \infty, \qquad \frac{14}{\infty} = 0, \qquad \infty + \infty = \infty$$

for $\lim_{n\to\infty}\left(\sqrt{n} - \frac{5n}{n-1}\right)$, etc., but be careful! We *cannot* say

$$\infty - \infty = 0 \qquad \text{or} \qquad \frac{\infty}{\infty} = 1$$

because, for example,

$$\lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}, \qquad \lim_{n\to\infty}\frac{2^n}{2n+1} = \infty, \qquad \lim_{n\to\infty}\frac{\sqrt{n}}{2n+1} = 0$$

are all "$\frac{\infty}{\infty}$".

There is no way to simplify $\frac{\infty}{\infty}$ that always works.

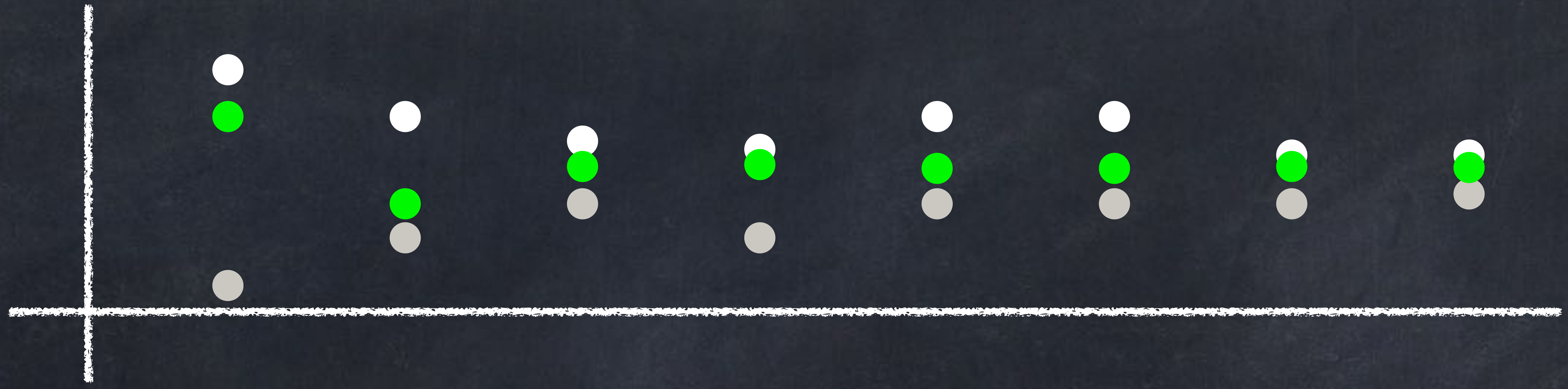
This is an example of an **indeterminate form**. Other indeterminate forms include

$$\infty - \infty, \qquad \frac{0}{0}, \qquad 0 \times \infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

Depending on what formulas are causing 0 or $\pm\infty$ to appear, limits with these patterns can have many different values.

More limit rules

The Squeeze Theorem: if $a_n \leq b_n \leq c_n$ for all $n > N$, and $\lim_{n\to\infty} a_n = L$, and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



The Comparison Test: if $a_n \leq b_n$ for all $n > N$ and $\lim_{n\to\infty} a_n = \infty$ then $\lim_{n\to\infty} b_n = \infty$.

(There is a similar rule about $-\infty$ too.)

Many limits can be calculated using the Squeeze Theorem, but finding (and proving!) useful inequalities can be difficult.

- Example 1 (good): $\lim_{n\to\infty} \frac{3\sin(n^5)}{n^2} = 0$ because $\frac{-3}{n^2} \leq \frac{3\sin(n^5)}{n^2} \leq \frac{3}{n^2}$ and we know $\lim_{n\to\infty} \frac{-3}{n^2} = 0$ and $\lim_{n\to\infty} \frac{3}{n^2} = 0$ from other rules.

- Example 2 (hard): $\lim_{n\to\infty} n^{1/n} = 1$ because $1 \leq n^{1/n} \leq \frac{\sqrt{n}+2}{\sqrt{n}}$ and we know $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} \frac{\sqrt{n}+2}{\sqrt{n}} = 1$ from other rules.

Functions

We have been talking about sequences, but for the the rest of the year we will be dealing with functions.

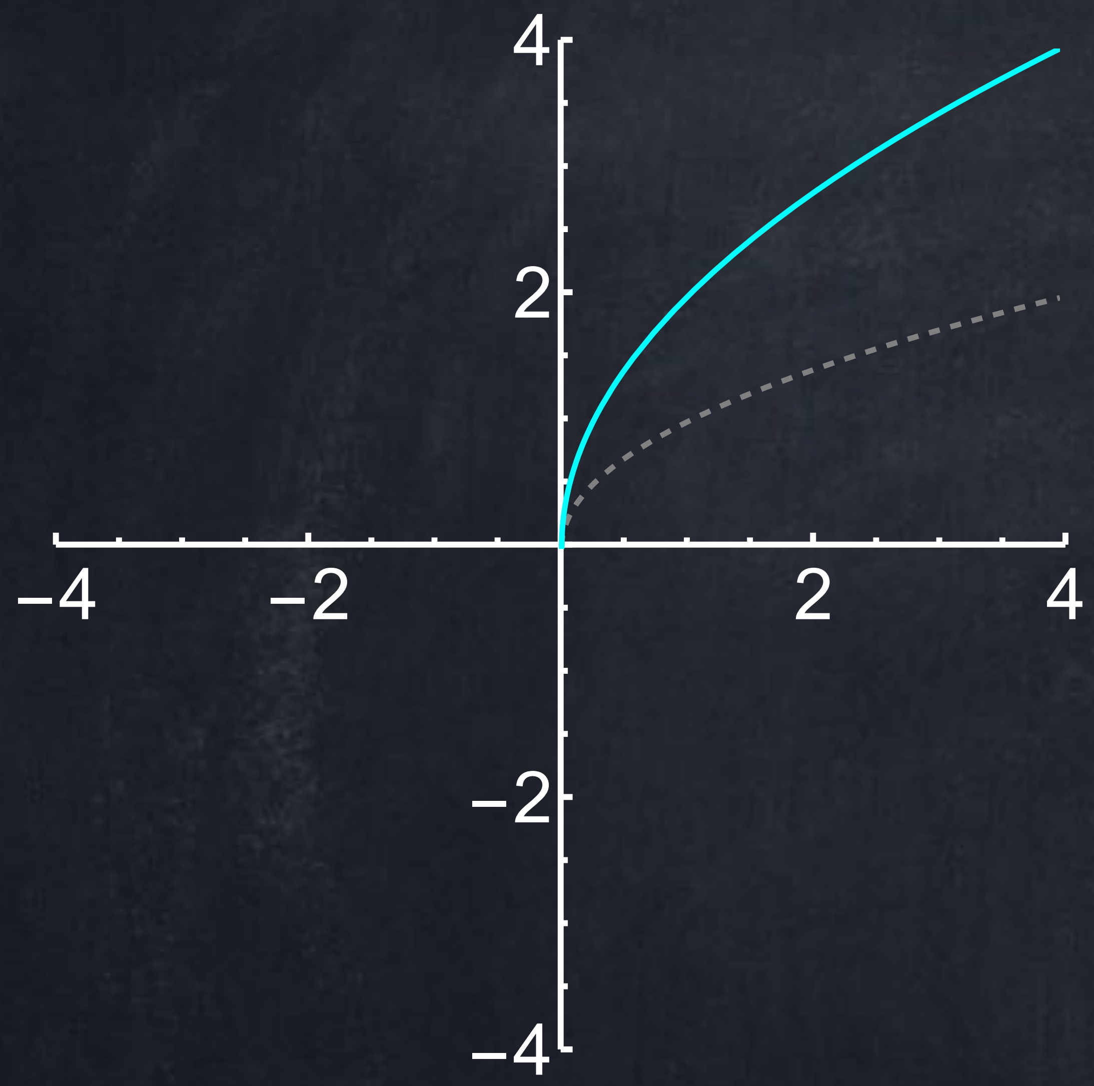
- $f(x) = 2\sqrt{x}$
- $g(x) = \quad x + 2$
- $f(x) = \ln(x) + 1$
- $f(t) = \cos(3t)$

You should be able to draw these graphs by hand already.

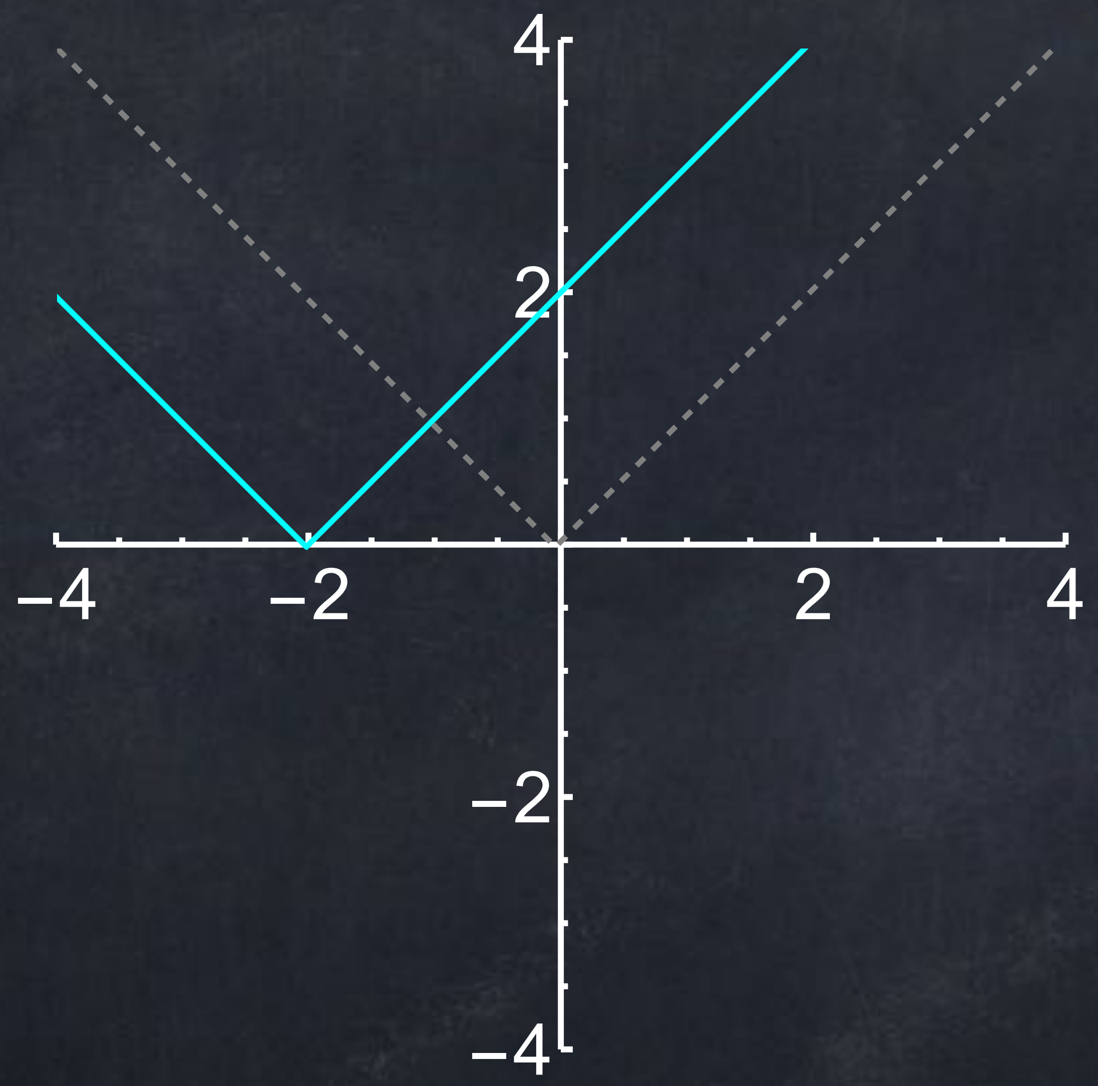
- $P(x) = x^3 - x$
- $r(x) = \dfrac{x+4}{x^2 - 2x}$
- $f(x) = \arctan(x)$

We will analyze these later.

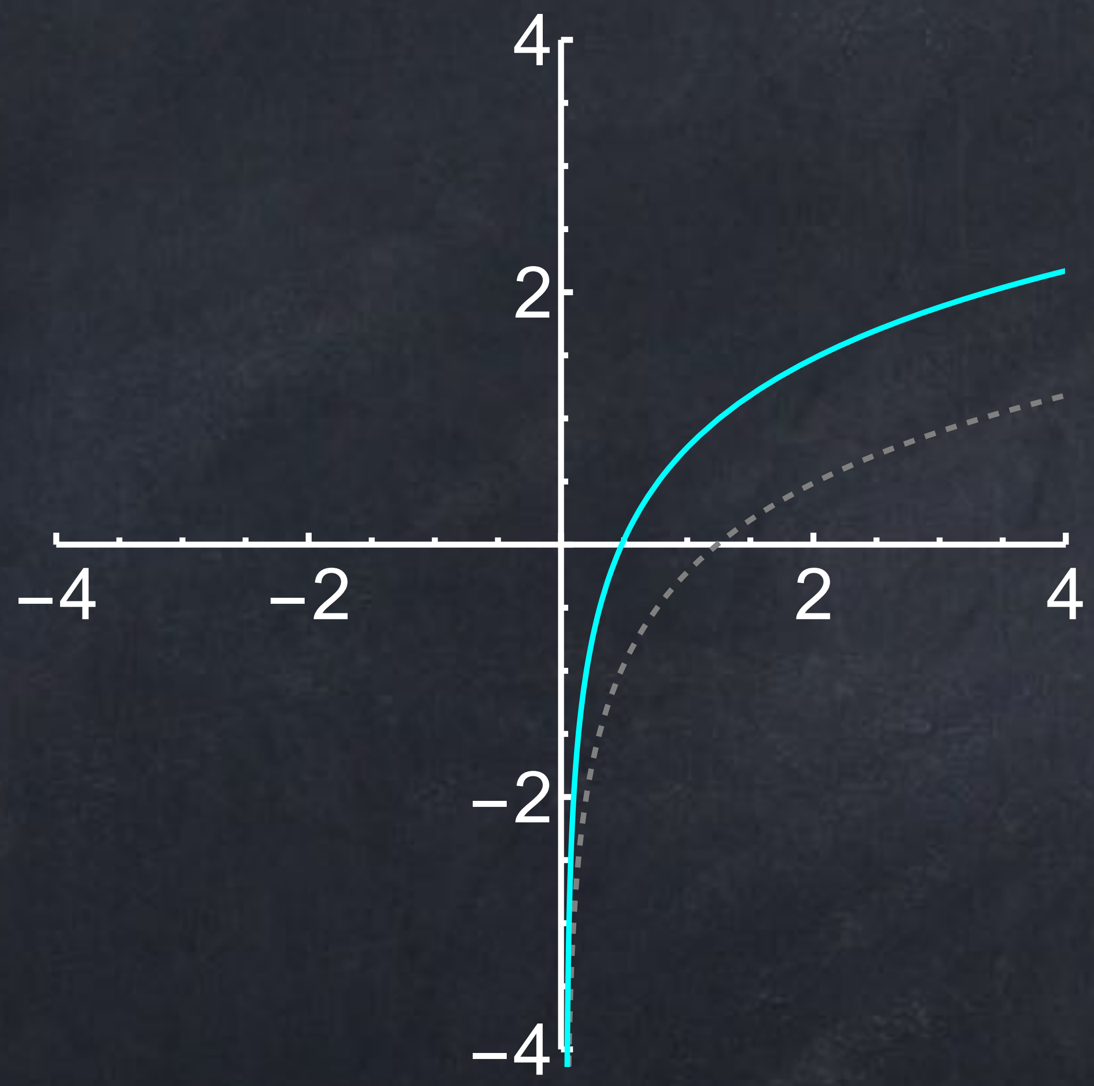
$$y = 2\sqrt{x}$$



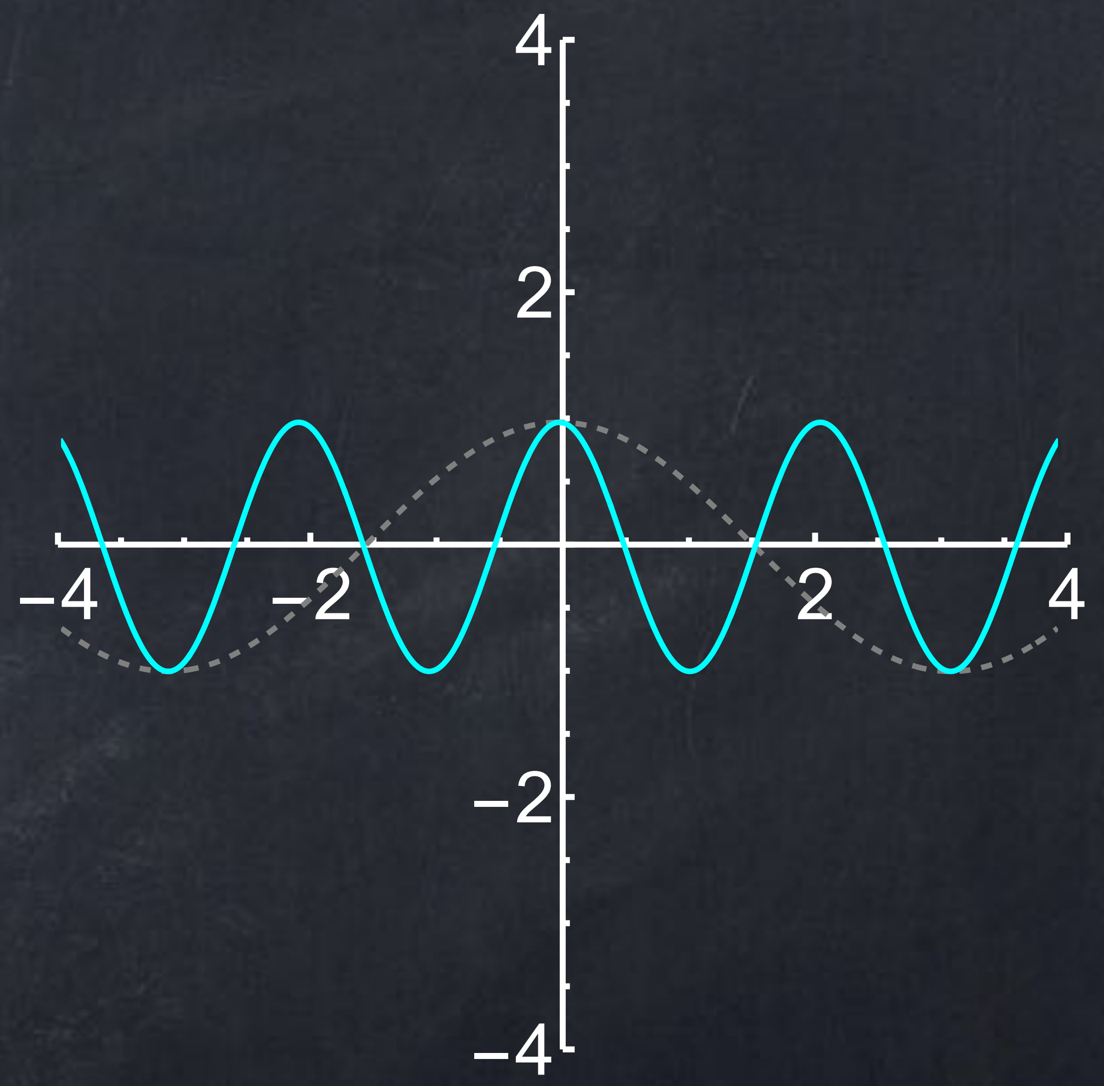
$$y = |x - 2|$$



$$y = \ln(x) + 1$$



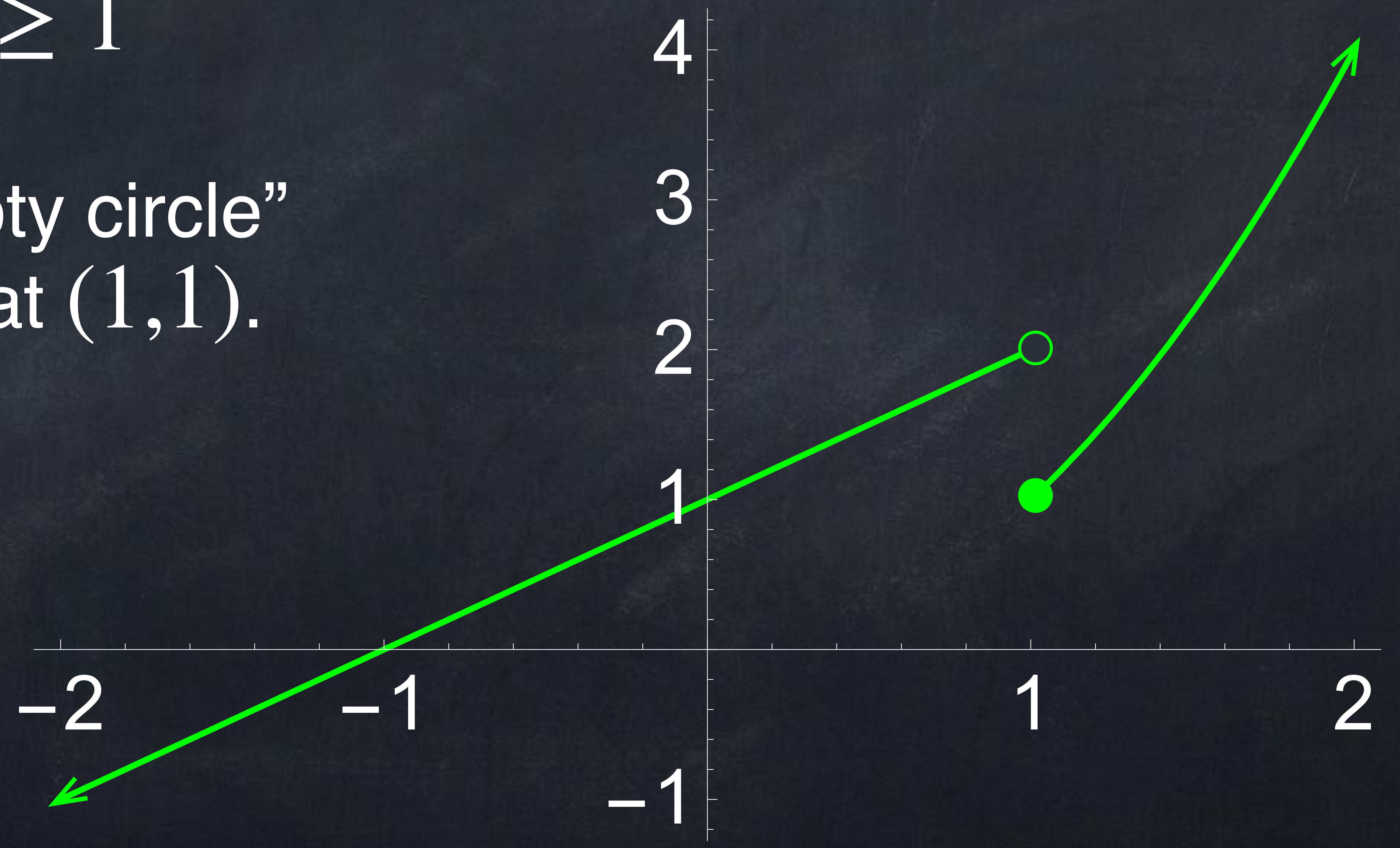
$$y = \cos(3t)$$



A **piecewise-defined function** uses different formulas for different inputs. We write these using a single large "curly bracket" ($\{$).

Example: $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}.$

Note the "open circle" ○ or "empty circle" at $(1,2)$ and the "filled circle" ● at $(1,1)$.



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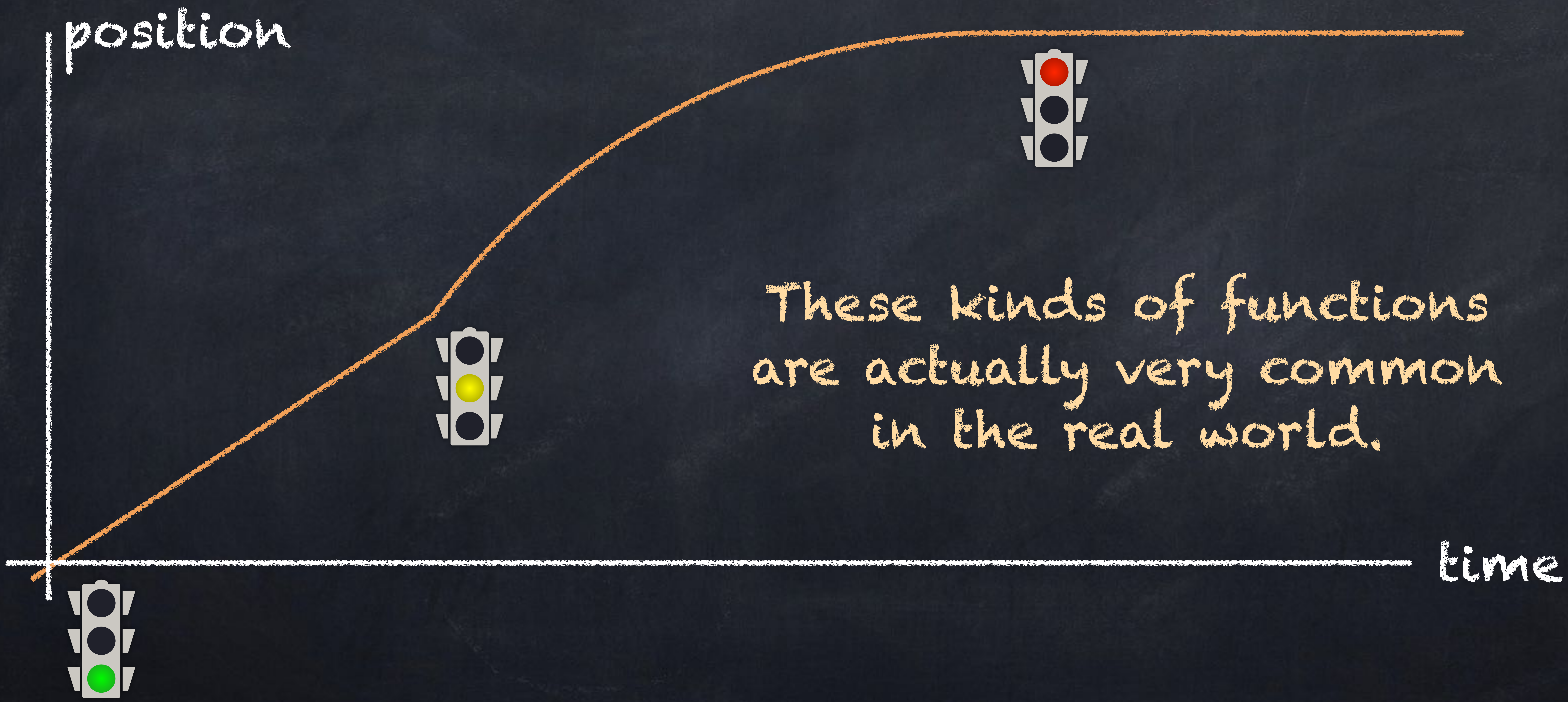


Limits as $x \to \pm\infty$

We can do limits with functions. "$\lim_{x\to\infty}$" is almost identical to "$\lim_{n\to\infty}$". The official definitions are

- $\lim_{n\to\infty} a_n = L$ means that for any $\varepsilon > 0$ there exists a N such that

  if $n \in \mathbb{N}$ and $n > N$ then $|a_n - L| < \varepsilon$.

- $\lim_{x\to\infty} f(x) = L$ means that for any $\varepsilon > 0$ there exists an X such that

  if $x > X$ then $|f(x) - L| < \varepsilon$.

There is also "$\lim_{x\to-\infty}$", but this is almost the same: $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x)$.

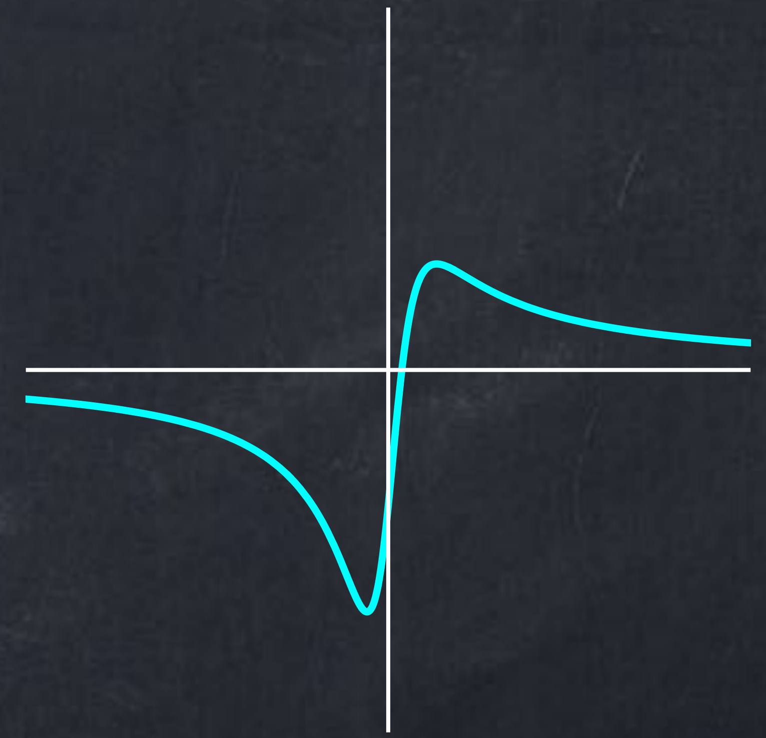
Limits as $x \to \pm\infty$

The line $y = c$ is a **horizontal asymptote** of the graph $y = f(x)$ if

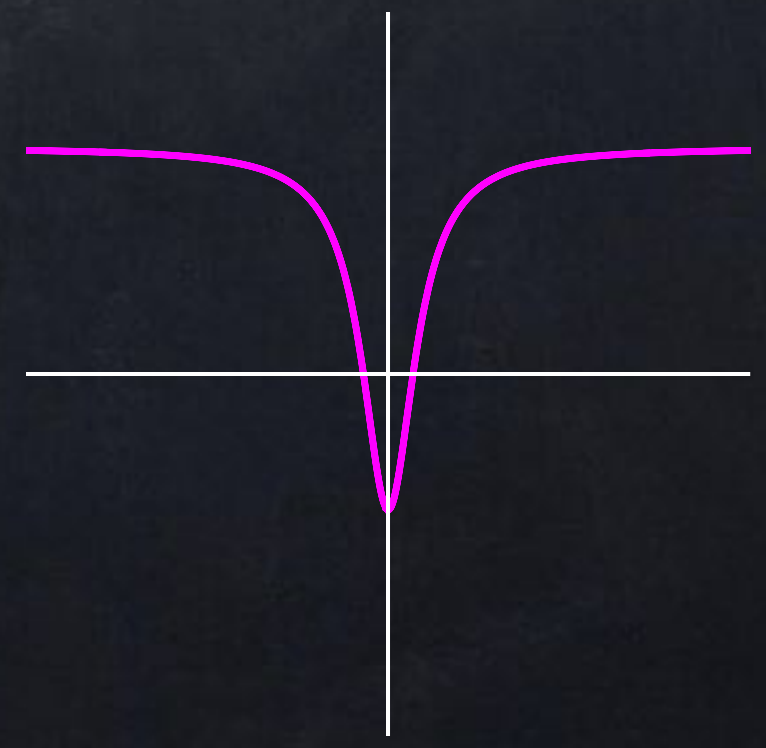
$$\lim_{x \to -\infty} f(x) = c \quad \text{or} \quad \lim_{x \to \infty} f(x) = c.$$

Examples:

- $f(x) = \dfrac{10x - 3}{2x^2 + 1}$ has a horizontal asymptote at $y = 0$.



- $f(x) = \dfrac{10x^2 - 3}{2x^2 + 1}$ has a horizontal asymptote at $y = 5$.



For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

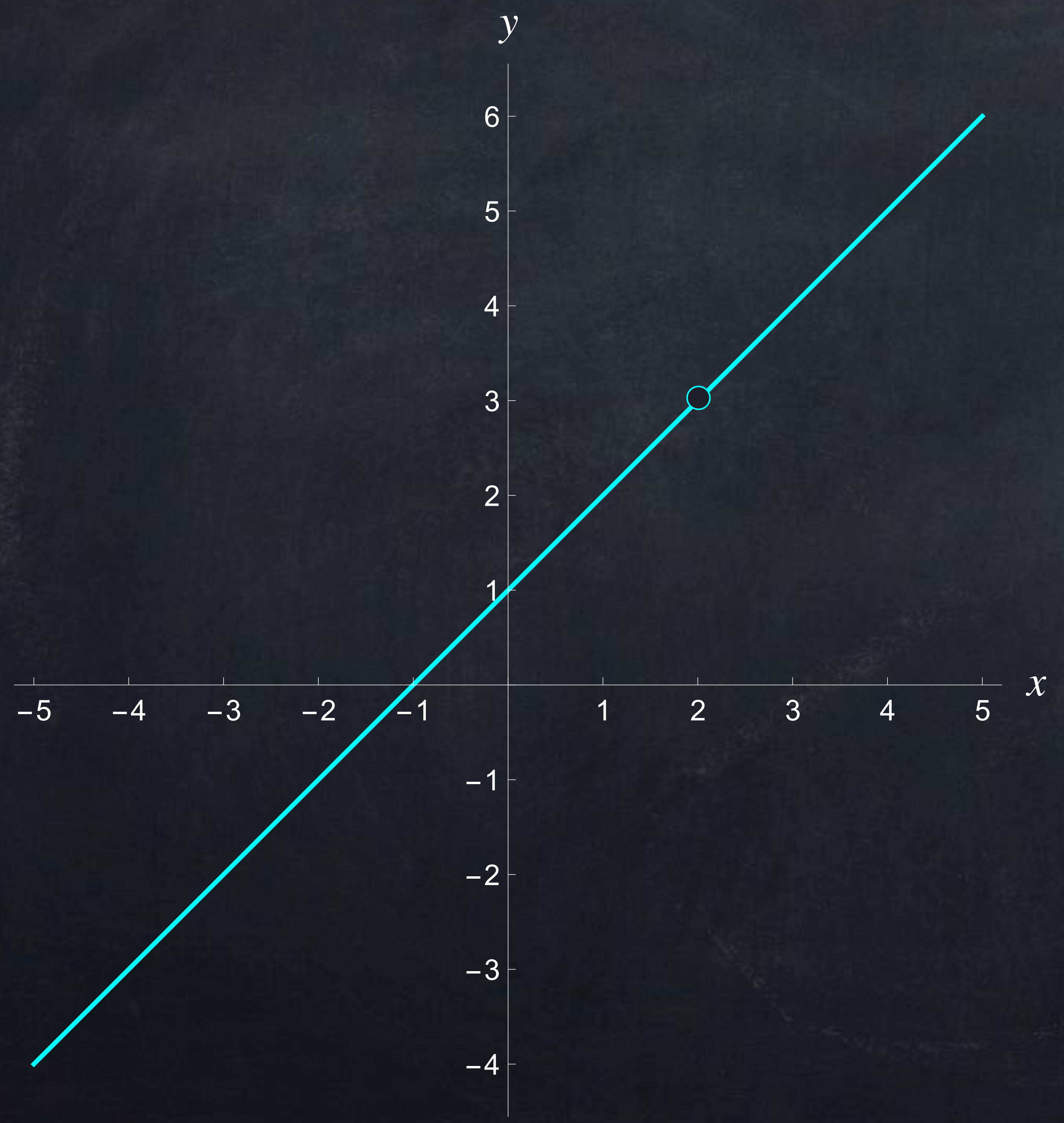
when $x = 2$,

$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0} = 🙁.$$

But if we look at the graph $y = \dfrac{x^2 - x - 2}{x - 2}$, we will be able to say more about $f(2)$.

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

$$\lim_{x \to 2} f(x) = 3$$

for this function.

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

we can also use a table of values to find $\lim_{x \to 2} f(x)$.

x	1.8	1.9	1.99	1.999	2.001	2.005	2.1
$f(x)$	2.8	2.9	2.99	2.999	3.001	3.005	3.1

These are very close to 3.

Note: this "limit" is about what happens when the input is CLOSE to a certain value but NOT exactly equal to it. We do NOT include x = 2 in this table.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

The equation above is said out loud as

"the limit as X goes to A of F of X equals L"

or

"the limit as X approaches A of F of X equals L".

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

Again there is an official definition using "ε" as any small value:

- $\lim_{x \to a} f(x) = L$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

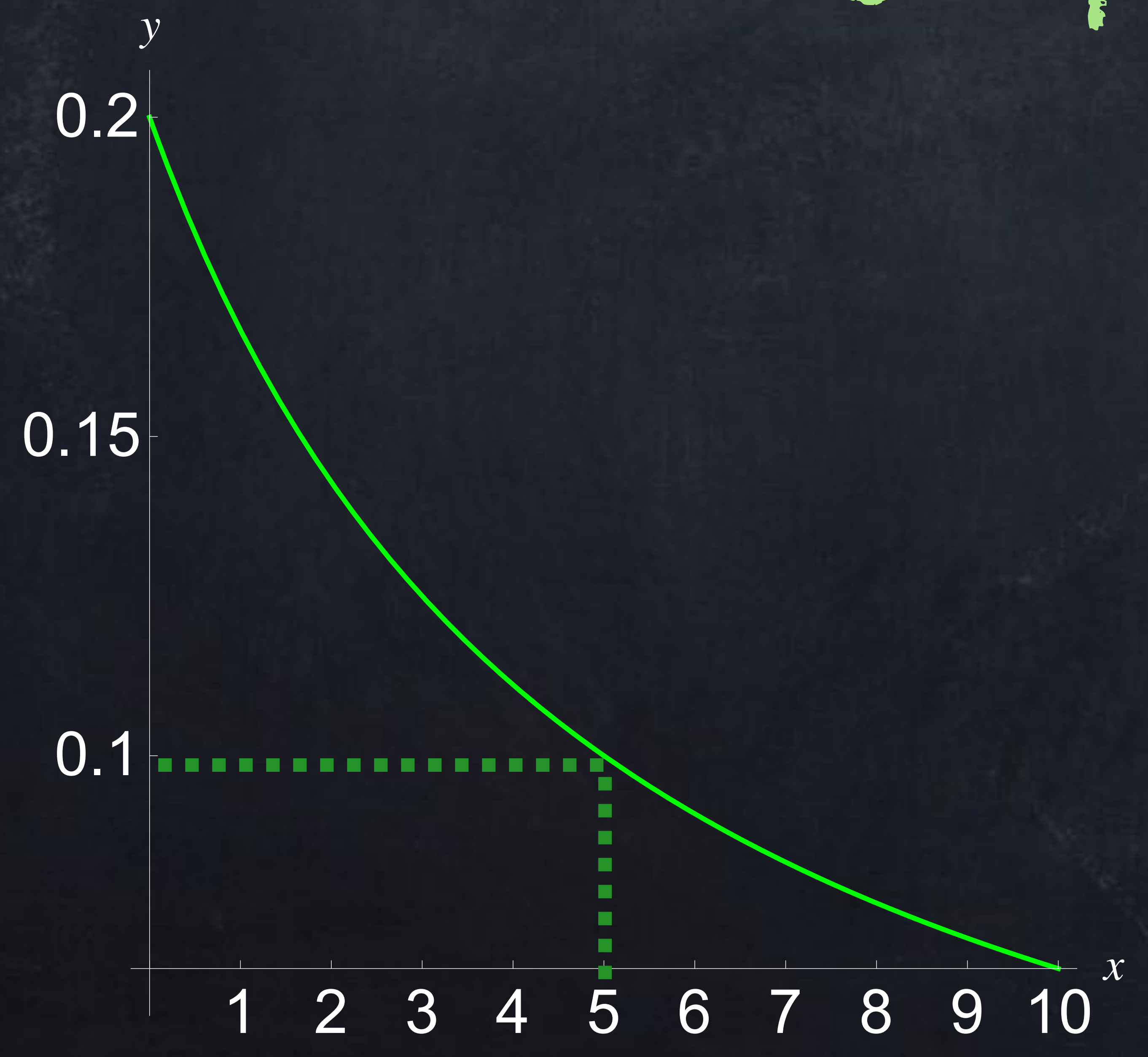
  if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Example: find $\lim_{x \to 5} \frac{x-5}{x^2-25}$.

Method 1: table

x	4.9	4.95	4.99	4.999	5.001	5.005	5.02	5.1
$f(x)$	0.10101	0.10050	0.10010	0.10001	0.09999	0.09995	0.09980	0.09901

Method 2: graph



Method 3: algebra

$\frac{x-5}{(x-5)(x+5)}$ simplifies* to $\frac{1}{x+5}$,

and when $x = 5$, we have $\frac{1}{(5)+5} = \frac{1}{10}$.

*Technically $\frac{x-5}{(x-5)(x+5)} = \frac{1}{x+5}$ requires $x \neq 5$, but since "$\lim_{x \to 5}$" is about when x is *near* 5, <u>not</u> exactly 5, this is okay.

Limit properties

For any numbers a and c,

- $\lim_{x \to a} c = c$ and
- $\lim_{x \to a} x = a$.

Examples: $\lim_{x \to 6}(27) = 27$ and $\lim_{x \to 6}(x) = 6$.

These should not be surprising.

Limit properties

If the limits all exist and are finite, then

- $$\lim_{x\to a}\Big(f(x)+g(x)\Big)=\Big(\lim_{x\to a}f(x)\Big)+\Big(\lim_{x\to a}g(x)\Big),$$
- $$\lim_{x\to a}\Big(f(x)\cdot g(x)\Big)=\Big(\lim_{x\to a}f(x)\Big)\Big(\lim_{x\to a}g(x)\Big),$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$$ if $\lim_{x\to a}g(x)\neq 0,$
- $$\lim_{x\to a}f\big(g(x)\big)=f\Big(\lim_{x\to a}g(x)\Big)$$ if f is a "nice" function.

with $g(x) = c$

- $$\lim_{x\to a}\Big(c\cdot f(x)\Big)=c\cdot\Big(\lim_{x\to a}f(x)\Big)$$

Later we will see exactly when $\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right)$ is allowed.

For now, it is enough to know that…

- any polynomial
  - This includes x^2.

$$\lim_{x \to a} \left(f(x)^2\right) = \left(\lim_{x \to a} f(x)\right)^2$$

- $\sqrt[n]{x}$
- $\sin(x)$ and $\cos(x)$
- e^x and a^x with $a > 0$
- $\ln(x)$ and $\log_b(x)$ with $b > 0$

can all be used safely in this limit rule.

*You might only be allowed to use $x \geq 0$ or $x > 0$ with these functions.

Example: Calculate $\lim_{x \to 3} x^2 - 15x + 9$ using the limit properties.

$$\lim_{x \to 3} x^2 - 15x + 9 = \left(\lim_{x \to 3} x^2\right) - \left(\lim_{x \to 3} 15x\right) + \left(\lim_{x \to 3} 9\right)$$

$$= \left(\lim_{x \to 3} x\right)^2 - 15\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 9\right)$$

$$= (3)^2 - 15 \cdot (3) + (9)$$

$$= -27$$

This is same as the value of $x^2 - 15x + 9$ itself when $x = 3$.

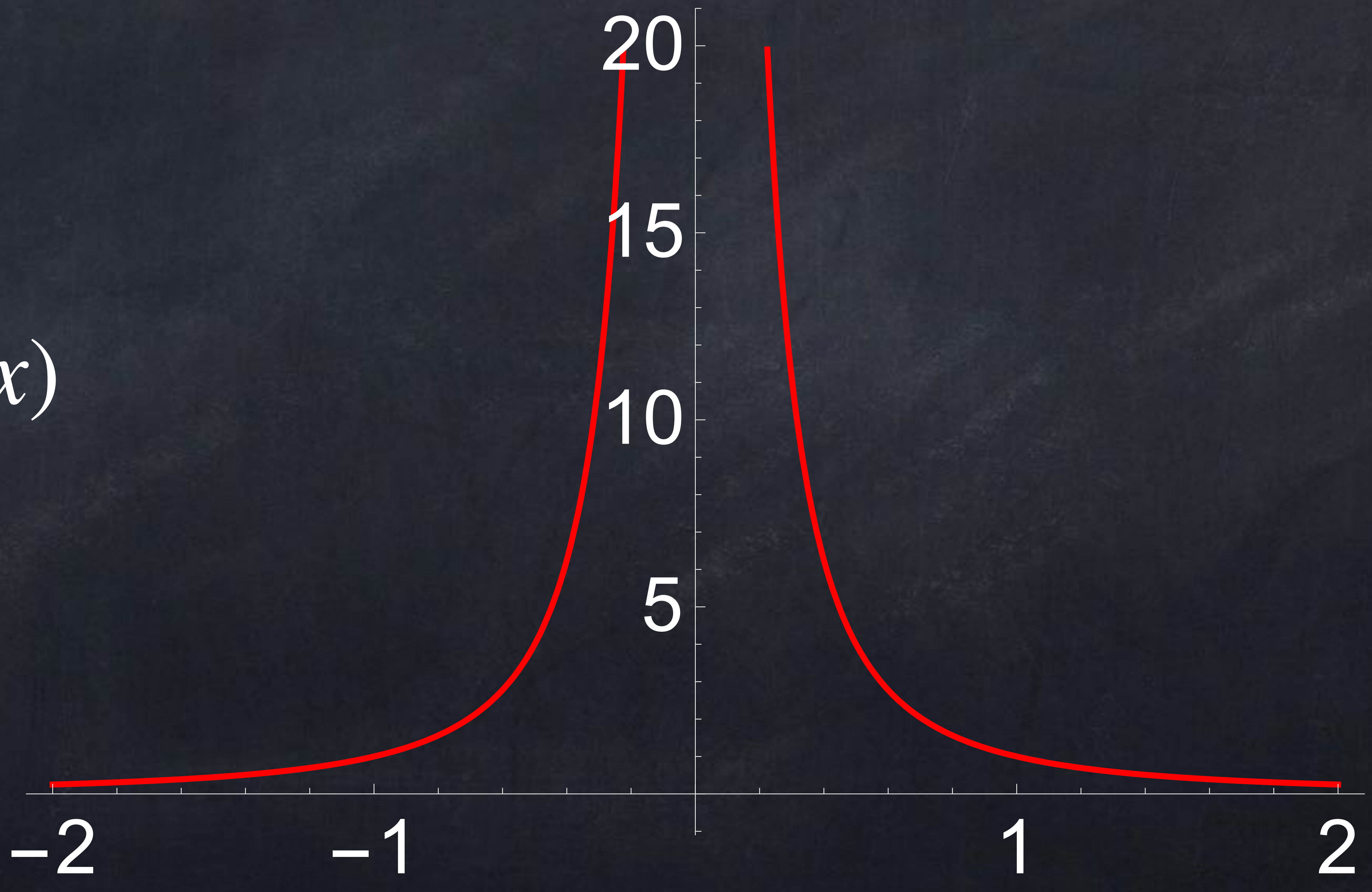
I will say more later about when we can find limits just by plugging in an x value.

Infinite Limits

Sometimes the limit as x approaches some finite point will be ∞ or $-\infty$.

For example, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

This means that for values of x very close to 0, the values of $f(x)$ are all extremely large.



Infinite Limits

Sometimes the limit as x approaches some finite point will be ∞ or $-\infty$.

For example, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Official definitions (we won't use these):

- $\lim_{x \to a} f(x) = \infty$ means that for any $M > 0$ there exists $\delta > 0$ such that

  if $|x - a| < \delta$ then $f(x) > M$.

- $\lim_{x \to a} f(x) = -\infty$ means that for any $M > 0$ there exists $\delta > 0$ such that

  if $|x - a| < \delta$ then $f(x) < -M$.

Some limit properties, such as

$$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big),$$

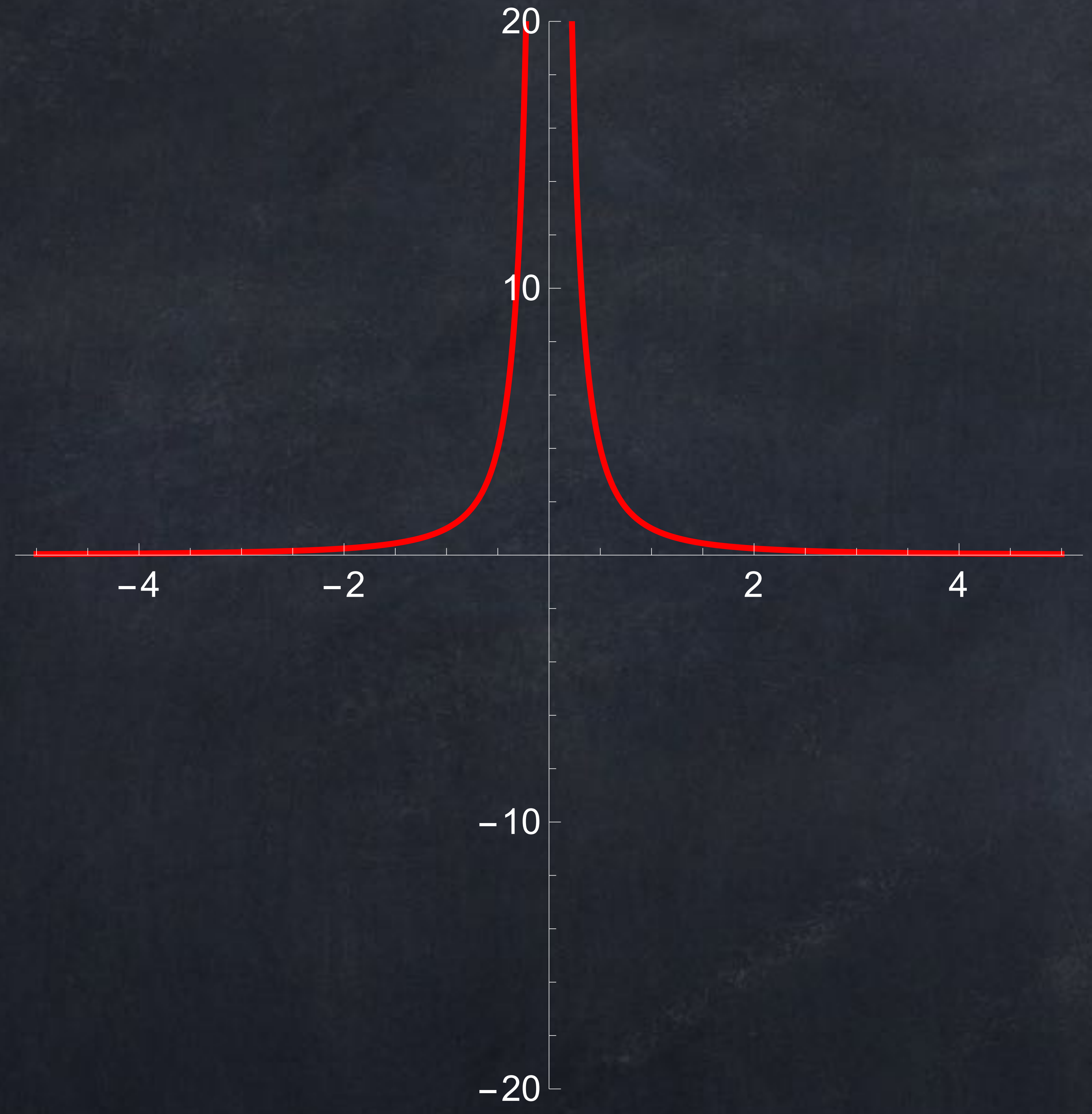
do not work with infinite limits.

Remember that $\infty - \infty$ and $\frac{\infty}{\infty}$ are indeterminate forms. We can ***not*** just say that "$\infty - \infty = 0$" because subtracting functions with infinite limits can give many different answers. Both

- $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^2}\right) = 0$
- $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^4}\right) = -\infty$

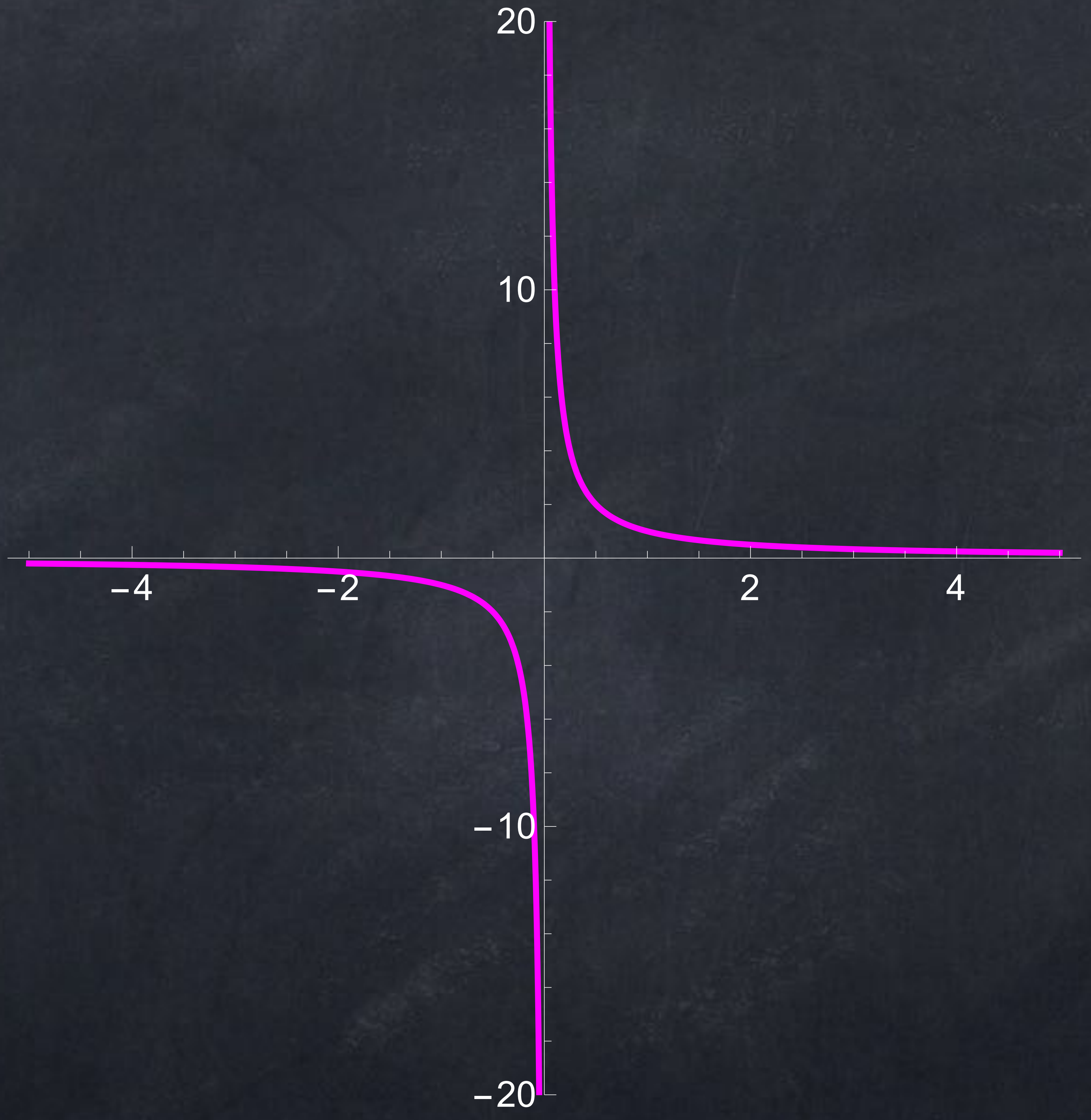
are "$\infty - \infty$" in some way.

$$y = \frac{1}{x^2}$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$y = \frac{1}{x}$$



$$\lim_{x \to 0} \frac{1}{x} \text{ doesn't exist}$$

One-sided Limits

We write

$$\lim_{x \to a^-} f(x)$$

for the "**limit as x approaches a from the left**" or "… from below". This means we only look at x values that are less than a.

Similarly,

$$\lim_{x \to a^+} f(x)$$

means the "**limit as x approaches a from the right**" or "… from above", where we only look at x values that are more than a.

One-sided limits

We write

$$\lim_{x \to a^-} f(x)$$

for the "limit as x approaches a from the left" or "… from below". This means we only look at x values that are less than a.

Example: $\lim_{x \to 0^-} x\sqrt{1+\frac{1}{x^2}} = -1.$

x	-0.1	-0.05	-0.01	-0.001	-0.0001
f(x)	-1.00499	-1.001249	-1.00005	-1.0000005	-1.00000001

One-sided Limits

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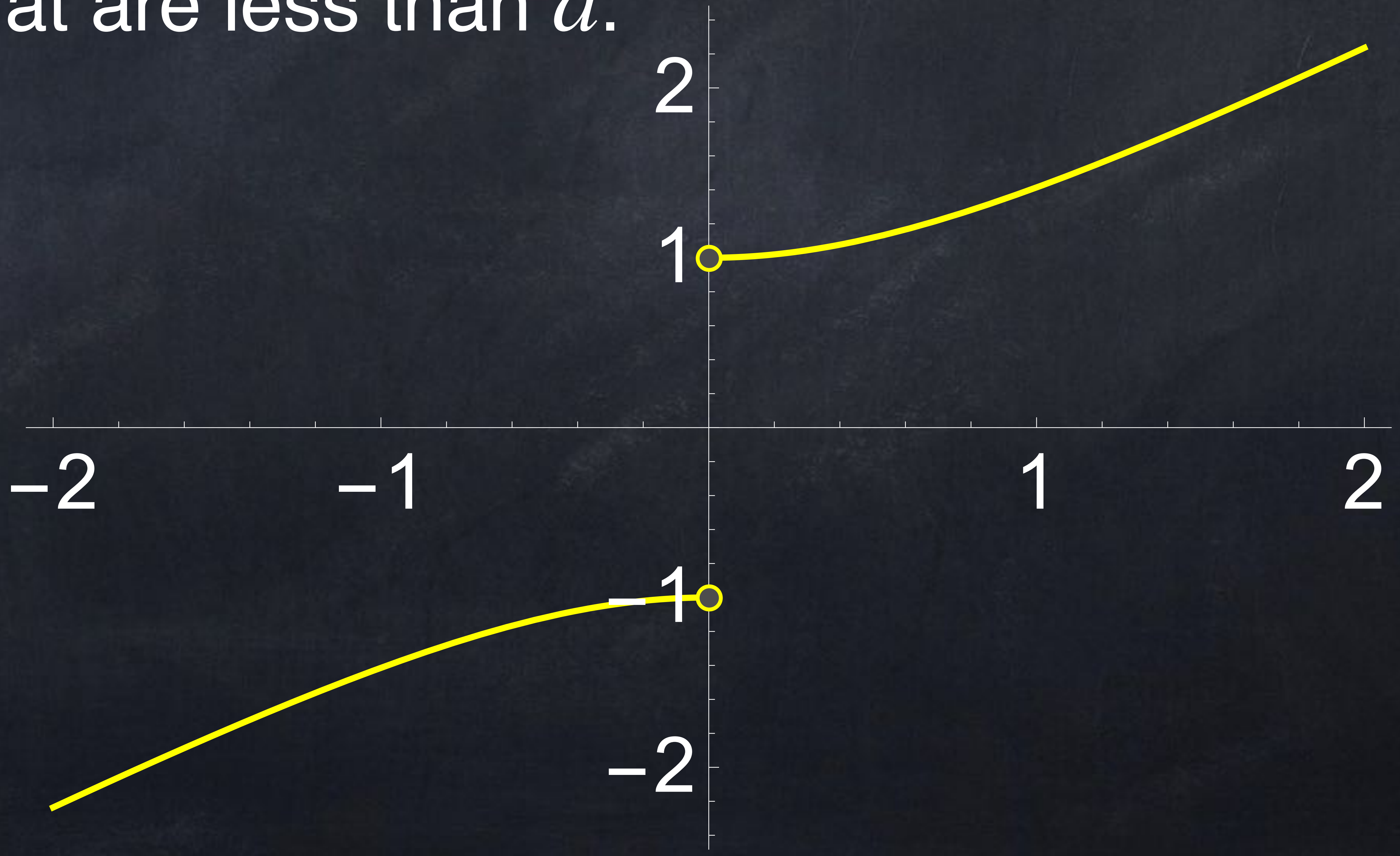
$$\lim_{x \to a^-} f(x)$$

for the "limit as x approaches a from the left" or "… from below". This means we only look at x values that are less than a.

Example: $$\lim_{x \to 0^-} x\sqrt{1+\frac{1}{x^2}} = -1.$$

$$\lim_{x \to 0^+} x\sqrt{1+\frac{1}{x^2}} = 1.$$



One-sided Limits

Note: writing

$$4^+$$

by itself does not mean anything (like $\sqrt{\ }$ or alone). This should only be written as part of a limit:

$$\lim_{x \to 4^+} f(x).$$

Some books use $\lim_{x \nearrow 4} f(x)$ and $\lim_{x \searrow 4} f(x)$ instead of $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

One-sided Limits

All of the limit rules for functions, such as

- $$\lim_{x\to a}\Big(f(x)+g(x)\Big)=\Big(\lim_{x\to a}f(x)\Big)+\Big(\lim_{x\to a}g(x)\Big),$$

can also be used with one-sided limits:

- $$\lim_{x\to a^-}\Big(f(x)+g(x)\Big)=\Big(\lim_{x\to a^-}f(x)\Big)+\Big(\lim_{x\to a^-}g(x)\Big),$$
- $$\lim_{x\to a^+}\Big(f(x)+g(x)\Big)=\Big(\lim_{x\to a^+}f(x)\Big)+\Big(\lim_{x\to a^+}g(x)\Big).$$

One-sided limits are related to standard limits in the following way:

If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ have different values, or if at least one of them does not exist, then $\lim_{x \to a} f(x)$ does not exist.

Logically, this also means that

- if $\lim_{x \to a} f(x)$ exists then $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal.